McGill University December 2011
Faculty of Engineering Final examination

Differential Equations for Engineers

Math 263

Wednesday, December 14, 2011

Time: 2pm-5pm

Examiner: Prof. J.J. Xu Associate Examiner: P. Reynolds

SOLUTIONS OF FINAL EXAMINATION

1. (a) (**8 points**) Solve the differential equation

$$2xyy' + x^2 = 3y^2 .$$

(b) (**2 points**) Give another method that we have learned of solving this equation. You may only briefly outline the second method without carrying it out in detail.

Solution:
Re-arranging, we have

$$x^2 - 3y^2 + 2xyy' = 0$$

which is not exact since $M_y = -6y$ and $N_x = 2y$, and these are not equal. However,

$$\frac{M_y - N_x}{N} = \frac{-8y}{2xy} = -\frac{4}{x}$$

which is a function of x alone. Thus we find an integrating factor $\mu(x)$ by solving

$$\mu'(x) = -\frac{4}{x}\mu(x)$$

which gives $\mu(x) = x^{-4}$. Thus multiplication of the original DE by this integrating factor gives the DE

$$\frac{1}{x^2} - 3\frac{y^2}{x^4} + 2\frac{y}{x^3}y' = 0$$

which is exact (as is easily checked). Thus we seek $F(x, y)$ such that

$$F_y = 2\frac{y}{x^3}\,, \quad F_x = \frac{1}{x^2} - 3\frac{y^2}{x^4}\,.$$

Thus $F(x, y) = \frac{y^2}{x^3} - \frac{1}{x}$ and so the solutions are given implicitly by

$$\frac{y^2}{x^3} - \frac{1}{x} = C\,.$$

(b) (**2 points**) There is another method, that we have learned, of solving this equation. Determine what this other method is and briefly outline it (you do not need to carry it out in detail).

Solution:
The equation can be rearranged as

$$y' = \frac{3y^2 - x^2}{2xy} = \frac{3}{2}\frac{y}{x} - \frac{1}{2}\frac{x}{y}$$

which is a homogeneous DE. Applying the substitution $u = y/x$ results in the DE

$$u' = \frac{1}{2x}\left(u - \frac{1}{u}\right)$$

which is separable. So solve it by integrating:

$$\int \frac{2u}{u^2-1}du = \int \frac{1}{x}dx$$

which gives

$$u^2 - 1 = Cx$$

or

$$\frac{y^2}{x^3} - \frac{1}{x} = C\,.$$

2. (**10 points**) Solve the differential equation

$$y' - (A\cos t + B)y + y^3 = 0\,.$$

Solution:
This is a Bernoulli DE. We apply the substitution $u = y^{1-3} = y^{-2}$, and assume $y \neq 0$:

$$\begin{aligned} u' &= -2y^{-3}y' \\ &= -2y^{-3}((A\cos t + B)y - y^3) \\ &= -2(A\cos t + B)u + 2 \end{aligned}$$

which is a first order nonhomogeneous DE for u. Writing it as

$$u' + 2(A\cos t + B)u = 2$$

we compute an integrating factor

$$\mu(t) = e^{\int 2A\cos t + 2B dt} = e^{2A\sin t + 2Bt}$$

and thus

$$e^{2A\sin t + 2Bt}u = 2\int e^{2A\sin t + 2Bt}dt + C$$

so

$$u = 2e^{-2A\sin t - 2Bt}\left(\int e^{2A\sin t + 2Bt}dt + C\right).$$

Since $y = u^{-1/2}$, we obtain the general solution as follows:

$$\begin{aligned} y &= \pm\left[2e^{-2A\sin t - 2Bt}\left(\int e^{2A\sin t + 2Bt} dt + C\right)\right]^{-1/2}; \\ y &= 0. \end{aligned}$$

3. (**15 points**) By using the *method of differential operators*, solve the Eq.

$$y^{(4)} - 2y'' + y = \mathrm{e}^t + \sin t.$$

1. Determine the annihilator of the inhomogeneous term and the form of particular solution;
2. Find the particular solution and the general solution for the equation;
3. Find the solution of the IVP with the IC's:

$$y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 1.$$

Solution:
1. The inhomogeneous term $b(t) = b_1(t) + b_2(t)$, where $b_1(x) = \mathrm{e}^t$, $b_2(t) = \sin t$, the corresponding annihilator is

$$Q(D) = Q_1(D) \circ Q_2(D)$$

where $Q_1(D) = (D-1)$, $Q_2(D) = (D^2+1)$.
2. Let the particular solution as $y_p = y_{p1} + y_{p2}$, where

$$P(D)\{y_{p1}\} = b_1(t), \quad P(D)\{y_{p2}\} = b_2(t),$$

and $P(D) = (D^4 - 2D^2 + 1) = (D-1)^2(D+1)^2$.
Note that the operators $P(D)$ $Q_1(D)$ have a common root $D = 1$, and the multiplicity this root in $P(D)$ is $m = 2$, while the operators $P(D)$ $Q_2(D)$ have no common root. Therefore, it can be deduced that the form of a particular solution is

$$y_{p1} = t^2 \mathrm{ker}\{\mathrm{Q_1(D)}\}, \quad y_{p2} = \mathrm{ker}\{\mathrm{Q_2(D)}\},$$

so that we have

$$y_p = B_2 t^2 \mathrm{e^t} + \mathrm{A}\cos\mathrm{t} + \mathrm{B}\sin\mathrm{t}.$$

Then it follows that

$$\begin{aligned} y_p' &= 2B_2 t\mathrm{e^t} + \mathrm{B_2 t^2 e^t} - \mathrm{A}\sin\mathrm{t} + \mathrm{B}\cos\mathrm{t}, \\ y_p'' &= 2B_2 t\mathrm{e^t} + \mathrm{2B_2te^t} + \mathrm{B_2t^2e^t} - \mathrm{A}\cos\mathrm{t} - \mathrm{B}\sin\mathrm{t}, \\ y_p''' &= 6B_2\mathrm{e^t} + \mathrm{3B_2te^t} + \mathrm{B_2t^2e^t} + \mathrm{A}\sin\mathrm{t} - \mathrm{B}\cos\mathrm{t}, \\ y_p^{(4)} &= 12B_2\mathrm{e^t} + \mathrm{4B_2te^t} + \mathrm{B_2t^2e^t} + \mathrm{A}\cos\mathrm{t} + \mathrm{B}\sin\mathrm{t}. \end{aligned}$$

Inserting them into the Eq., we have

$$8B_2\mathrm{e^t} + 4\mathrm{A}\cos\mathrm{t} + 4\mathrm{B}\sin\mathrm{t} = \mathrm{e^t} + \sin\mathrm{t}.$$

Equalizing two sides, we get

$$8B_2 = 1, A = 0, 4, B = 1.$$

Then

$$B_2 = \frac{1}{8}, B = \frac{1}{4},$$

here B_0 , B_1 are taken arbitrarily. When we take both B_0 and B_1 as zero here, the particular solution is

$$y_p = \frac{1}{8}t^2\mathrm{e}^{\mathrm{t}} + \frac{1}{4}\sin\mathrm{t}.$$

The general solution for the equation is expressed as

$$y = (C_0 + C_1 t)\mathrm{e}^{\mathrm{t}} + (\mathrm{D}_0 + \mathrm{D}_1\mathrm{t})\mathrm{e}^{-\mathrm{t}} + \frac{1}{8}\mathrm{t}^2\mathrm{e}^{\mathrm{t}} + \frac{1}{4}\sin\mathrm{t},$$

where C_0, C_1, D_0, D_1 are arbitrary constants.

3. With the given initial conditions, we get

$$C_0 + D_0 = 0,$$

$$C_0 + C_1 + D_0 - D_1 + \frac{1}{4} = 1,$$

$$C_0 + 2C_1 + D_0 - 2D_1 + \frac{1}{4} = 0,$$

$$C_0 + 3C_1 - D_0 + 3D_1 + \frac{1}{2} = 0.$$

We solve the associated equations to find the particular solution:

$$C_0 = \frac{7}{16}, C_1 = -\frac{1}{8}, D_0 = -\frac{7}{16}, D_1 = 0.$$

4.(**10 points**) By changing variables and using the *differential operator method*, find the general solution of the following Eq:

$$x^3y''' + x^2y'' + \frac{1}{4}xy' - \frac{1}{4}y = 0.$$

Solution:

Let $x = \pm\mathrm{e}^{\mathrm{s}}$, $y(x) = \tilde{y}(s)$ $D = \frac{d}{ds}$, it is derived that

$$xy' = D\tilde{y}, x^2y'' = D(D-1)\tilde{y}, x^3y''' = D(D-1)(D-2)\tilde{y}.$$

Substituting them into the Eq., we have

$$P(D)\tilde{y} = \left(D(D-1)(D-2) + D(D-1) + \frac{1}{4}D - \frac{1}{4}\right)\tilde{y} = 0.$$

The polynomial $P(D)$has three roots

$$D = 1, \frac{1}{2}, \frac{1}{2}.$$

So, we can write the general solution for the equation

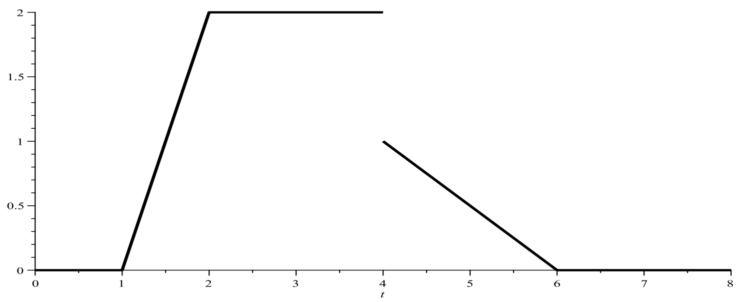
$$\tilde{y} = C_1\mathrm{e}^{\mathrm{s}} + (\mathrm{C}_2 + \mathrm{C}_3\mathrm{s})\mathrm{e}^{\frac{1}{2}\mathrm{s}}.$$

Returning to the variable x , we get

$$y = C_1x + C_2\sqrt{x} + C_3\sqrt{x}\ln|x|,$$

where C_1, C_2, C_3 are arbitrary constants.

5. (**10 points**) Compute the Laplace transform of the function $f(t)$ whose graph is pictured.



Solution:
The function $f(t)$ has the following formula:

$$f(t) = \begin{cases} 0 & 0 \le t < 1 \\ 2t-2 & 1 \le t < 2 \\ 2 & 2 \le t < 4 \\ -\frac{t}{2}+3 & 4 \le t < 6 \\ 0 & t \ge 6 \end{cases}$$

which is expressed in terms of the unit step function as follows:

$$\begin{aligned} f(t) &= [u_1(t)-u_2(t)](2t-2) + [u_2(t)-u_4(t)]2 + [u_4(t)-u_6(t)](-t/2+3) \\ &= 2u_1(t)(t-1) - 2u_2(t)(t-2) - \tfrac{1}{2}(t-2)u_4(t) + \tfrac{1}{2}(t-6)u_6(t) \\ &= 2u_1(t)(t-1) - 2u_2(t)(t-2) - \tfrac{1}{2}u_4(t)((t-4)+2) + \tfrac{1}{2}u_6(t)(t-6) \end{aligned}$$

and so the Laplace transform is

$$\begin{aligned} \mathcal{L}\{f(t)\} &= 2\mathcal{L}\{u_1(t)(t-1)\} - 2\mathcal{L}\{u_2(t)(t-2)\} - \tfrac{1}{2}\mathcal{L}\{(u_4(t)(t-4)+2)\} + \tfrac{1}{2}\mathcal{L}\{u_6(t)(t-6)\} \\ &= 2e^{-s}\mathcal{L}\{t\} - 2e^{-2s}\mathcal{L}\{t\} - \tfrac{1}{2}e^{-4s}\mathcal{L}\{t+2\} + \tfrac{1}{2}e^{-6s}\mathcal{L}\{t\} \\ &= 2e^{-s}\frac{1}{s^2} - 2e^{-2s}\frac{1}{s^2} - \tfrac{1}{2}e^{-4s}\left(\frac{1}{s^2}+\frac{2}{s}\right) + \tfrac{1}{2}e^{-6s}\frac{1}{s^2} \\ &= \frac{1}{2s^2}\left(4e^{-s} - 4e^{-2s} - e^{-4s}(1+2s) + e^{-6s}\right) \end{aligned}$$

6. (**15 points**) Solve the following initial value problem:

$$y'' + 3y' + 2y = \delta(t-5) + u_{10}(t), \qquad y(0) = 0, \quad y'(0) = \tfrac{1}{2}.$$

Solution:
Let us take the Laplace transform of both sides of the DE, denoting $\mathcal{L}\{y(t)\}$ as $Y(s)$:

$$\begin{aligned} &\mathcal{L}\{y'' + 3y' + 2y\} = \{\delta(t-5) + u_{10}(t)\} \\ &s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) + 2Y(s) = e^{-5s} + \frac{e^{-10s}}{s} \\ &(s^2+3s+2)Y(s) = e^{-5s} + \frac{e^{-10s}}{s} + \frac{1}{2} \\ &Y(s) = \frac{e^{-5s}}{(s+1)(s+2)} + \frac{e^{-10s}}{s(s+1)(s+2)} + \frac{1}{2(s+1)(s+2)} \end{aligned}$$

Now using partial fractions we find

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

and

$$\frac{1}{s(s+1)(s+2)} = \frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}\,.$$

Thus the last term of $Y(s)$ above is

$$\frac{1}{2(s+1)(s+2)} = \frac{1}{2}\left(\frac{1}{s+1} - \frac{1}{s+2}\right) = \frac{1}{2}\mathcal{L}\{e^{-t} - e^{-2t}\}$$

and the first term of $Y(s)$ above is

$$\frac{e^{-5s}}{(s+1)(s+2)} = e^{-5s}\mathcal{L}\{e^{-t} - e^{-2t}\} = \mathcal{L}\{u_5(t)(e^{-(t-5)} - e^{-2(t-5)})\}\,.$$

Finally, the second term of $Y(s)$ above is

$$\frac{e^{-10s}}{s(s+1)(s+2)} = e^{-10s}\mathcal{L}\{\tfrac{1}{2} - e^{-t} + \tfrac{1}{2}e^{-2t}\} = \mathcal{L}\{u_{10}(t)(\tfrac{1}{2} - e^{-(t-10)} + \tfrac{1}{2}e^{-2(t-10)}\}\,.$$

And so:

$$y(t) = \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t} + u_5(t)(e^{-(t-5)} - e^{-2(t-5)}) + u_{10}(t)(\tfrac{1}{2} - e^{-(t-10)} + \tfrac{1}{2}e^{-2(t-10)})\,.$$

7.(**15 points**) Given Eq.:

$$x^2y'' + 3xy' + (1+x)y = 0, \quad (x > 0).$$

1. Show that $x = 0$ is a regular singular points, derive the corresponding *indicial equation* and give the form of the series solutions near $x = 0$;
2. Derive the *recurrence formula* of the coefficients and show the **first non-zero three terms** of two linear independent solutions: $\{y_1(x), y_2(x)\}$.

   (Hit: You may find $y_1(x)$ first, then find the $y_2(x)$, by substituting the form of the solution into Eq.)

Solution:

1. From EQ., we get $p(x) = \frac{3}{x}$, $q(x) = \frac{1+x}{x^2}$; since

   $$xp(x) = 3,\ x^2q(x) = 1 + x,$$

   then

   $$p_0 = 3,\ q_0 = 1,\ q_1 = 1,$$

   others are all zero are analytic, then $x = 0$ is a regular singular point of the equation.

2. The indicial EQ. and the exponent at the singular point x = 0.

$$y = x^r \sum_{n=0}^{\infty} a_n x^n,$$

The indicial EQ. is

$$r(r-1) + 3r + 1 = 0,$$

then

$$r = -1, -1.$$

The form of the series solutions near $x = 0$ is

$$y_1 = x^{-1} \sum_{n=0}^{\infty} a_n(-1) x^n,$$

$$y_2 = y_1 \ln x + x^{-1} \sum_{n=0}^{\infty} a_n'(-1) x^n,$$

where $a_0'(0) = 0$.

3. Substituting $y = x^r \sum\limits_{n=0}^{\infty} a_n x^n$ into EQ., we get the recurrence formula

$$a_n(r) = \frac{-1}{(n+r+1)^2} a_{n-1}(r),$$

with $a_0 = 1$, then

$$a_1(r) = \frac{-1}{(2+r)^2} a_0, \ a_2(r) = \frac{(-1)}{(3+r)^2} a_1 = \frac{(-1)^2}{(2+r)^2(3+r)^2},$$

$$a_3(r) = \frac{-1}{(4+r)^2} a_2 = \frac{(-1)^3}{(2+r)^2(3+r)^2(4+r)^2}, \cdots$$

$$a_n(r) = \frac{-1}{(4+r)^2} a_2 = \frac{(-1)^n}{(2+r)^2(3+r)^2(4+r)^2 \cdots (n+1+r)^2}.$$

So we get

$$a_n(-1) = \frac{(-1)^n}{1^2 2^2 3^2 4^2 \cdots n^2} = \frac{(-1)^n}{(n!)^2}.$$

Thus

$$y_1 = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} x^n = \frac{1}{x} - 1 + \frac{1}{4} x + \cdots.$$

Taking the derivative of $a_n(r)$, we have

$$a_n'(r) = -2 \left(\frac{1}{(2+r)} + \frac{1}{(3+r)} + \cdots + \frac{1}{(n+1+r)} \right) a_{n-1}(r).$$

$$a_n'(-1) = (-1)^{n+1} \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} \right) \frac{2}{(n!)^2},$$

then the other linearly independent solution is

$$y_2 = \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}x^n\ln x - \frac{2}{x}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}\right)x^n$$
$$= \frac{1}{x}\ln x - \ln x + \frac{1}{4}x\ln x + \cdots + 2 - \frac{3}{4}x + \frac{11}{108}x^2 + \cdots .$$

8.(**15 points**) Given Eq:

$$\left(\cos x - \frac{1}{2}\right)y'' + \alpha y' - \frac{1}{(x-\frac{7\pi}{3})}y = 0,\ (x>0).$$

1. Find all the regular singular points: $x_0 < x_1 < x_2 < \cdots$; determine the indicial equation and the exponents corresponding to the first two smallest singular points $x = x_0, x_1$.
2. Determine the **forms** of two linear independent solutions near the smallest regular singular point x_0, corresponding to different values of α.
3. Determine the range of the parameter α, such that all solutions of the Eq. near the smallest regular singular point x_0 are bounded.

Solution:

1. It is seen from EQ. that $p(x) = \frac{\alpha}{(\cos x - \frac{1}{2})}$, $q(x) = \frac{-1}{(\cos x-\frac{1}{2})(x-\frac{7\pi}{3})}$; since

$$\left[x - \left(2k\pi + \frac{\pi}{3}\right)\right]p(x) \to -\frac{2\alpha}{\sqrt{3}} = p_0, \quad \text{as } x \to (2k\pi + \frac{\pi}{3})$$
$$\left[x - (2k\pi + \frac{5\pi}{3})\right]p(x) \to \frac{2\alpha}{\sqrt{3}} = p_0, \quad \text{as } x \to (2k\pi + \frac{5\pi}{3})$$

and for $k=1$,

$$\left[x - \frac{7\pi}{3}\right]^2 q(x) \to \frac{2}{\sqrt{3}} = q_0;$$

for $k \neq 1$,

$$\left[x - (2k\pi + \frac{\pi}{3})\right]^2 q(x) \to 0 = q_0,$$

for both $k=1$ and $k\neq 1$,

$$\left[x - (2k\pi + \frac{5\pi}{3})\right]^2 q(x) \to 0 = q_0,$$

So $x = (2k\pi + \frac{\pi}{3}), (2k\pi + \frac{5\pi}{3}), (k \geq 0, k \in Z)$ are all the regular singular points of the EQ.

Therefore, first two smallest regular points are: $x_0 = \pi/3,\ \ x_1 = 5\pi/3$.

- At the regular singular point $x_0 = \frac{\pi}{3}$, we get

$$p_0 = -\frac{2\alpha}{\sqrt{3}},\ q_0 = 0,$$

then the indicial equation is

$$r(r-1) - \frac{2\alpha}{\sqrt{3}}r = 0,$$

hence the exponents corresponding to the singular point $x_0 = \frac{\pi}{3}$ are

$$r_1 = 0,\ r_2 = 1 + \frac{2\alpha}{\sqrt{3}};$$

- At the regular singular point $x_1 = \frac{5\pi}{3}$, we get

$$p_0 = \frac{2\alpha}{\sqrt{3}},\ q_0 = 0,$$

then the indicial equation is

$$r(r-1) + \frac{2\alpha}{\sqrt{3}} r = 0,$$

hence the exponents corresponding to the singular point $x_1 = \frac{5\pi}{3}$ are

$$r_1 = 0,\ r_2 = 1 - \frac{2\alpha}{\sqrt{3}}.$$

2. Determine the forms of two linear independent solutions near the smallest regular singular point $x_0 = \frac{\pi}{3}$, corresponding to different values of α . From two roots, $r_{1,2} = 0; \left(1 + \frac{2\alpha}{\sqrt{3}}\right)$, we get that

   (a) when $\alpha \neq -\frac{\sqrt{3}}{2}$, $r_1 - r_2 \neq N$, we get

$$y_1 = |x - x_0|^{r_1} \sum_{n=0}^{\infty} a_n(r_1)(x - x_0)^n,$$

$$y_2 = |x - x_0|^{r_2} \sum_{n=0}^{\infty} a_n(r_2)(x - x_0)^n;$$

   (b) when $\alpha = -\frac{\sqrt{3}}{2}$, $r_1 = r_2 = 0$, we get

$$y_1 = \sum_{n=0}^{\infty} a_n(r_1)(x - x_0)^n,$$

$$y_2 = y_1 \ln|x - x_0| + \sum_{n=1}^{\infty} a_n'(r_2)(x - x_0)^n;$$

   (c) when $r_1 - r_2 = N, (N = 1, 2, \cdots)$, we get

$$y_1 = |x - x_0|^{r_1} \sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

$$y_2 = a y_1 \ln|x - x_0| + |x - x_0|^{r_2} \sum_{n=1}^{\infty} \tilde{a}_n(x - x_0)^n;$$

3. Since any solution of the linear Eqs. can be expressed as a combination of the two linearly independent solution, we only need analyze the basic solutions.

- If $\alpha < -\frac{\sqrt{3}}{2}$, but $\alpha \neq -N\frac{\sqrt{3}}{2}$, we have $r_1 = 0, r_2 < 0$, then

$$y_1 = O(1)$$

y_1 is always bounded, and

$$y_2 \sim |x - x_0|^{r_2} \to \infty, \quad \text{as} \quad x \to x_0,$$

so that, y_2 is unbounded. Therefore, in the range

$$\alpha < -\frac{\sqrt{3}}{2},$$

some solutions are bounded, some solutions are unbounded.

- If $\alpha > -\frac{\sqrt{3}}{2}$, but $\alpha \neq N\frac{\sqrt{3}}{2}$, we have $r_2 = 0, r_1 > 0$, then

$$y_2 = O(1),$$

y_2 is a bounded, and

$$y_1 \sim |x - x_0|^{r_1} \to 0, \quad \text{as} \quad x_0 \to 0,$$

y_1 is also bounded. So that all solutions are bounded.

- If $\alpha = -\frac{\sqrt{3}}{2}$, $r_1 = r_2 = 0$, then $y_1 = O(1)$ and y_2 is unbounded. Therefore, some solution bounded, some solutions are unbounded.

- If $\alpha = N\frac{\sqrt{3}}{2}$, $r_2 = 0, r_1 = N, (N = 0, 1, 2, \cdots)$, we get

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n,$$

$$y_2 = a y_1 \ln|x - x_0| + |x - x_0|^{r_2} \sum_{n=1}^{\infty} \tilde{a}_n (x - x_0)^n;$$

Both y_1 and y_2 are bounded, so that all solutions are bounded.

- when $\alpha = -N\frac{\sqrt{3}}{2}$, $r_1 = 0, r_2 = -N, (N = 1, 2, \cdots)$, we get

$$y_1 = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

$$y_2 = a y_1 \ln|x - x_0| + |x - x_0|^{r_2} \sum_{n=1}^{\infty} \tilde{a}_n (x - x_0)^n;$$

Then y_1 is bounded, while y_2 are unbounded, so that some solutions are bounded, some solutions are unbounded.